CONTACT PROBLEM FOR A HALF-PLANE WITH ELASTIC REINFORCEMENT

PMM Vol. 32, №4, 1968, pp. 632-646

N. Kh. ARUTIUNIAN
(Erevan)

(Received April 9, 1968)

A number of papers are devoted to the investigation of the state of stress in a half-plane or in a plate with elastic reinforcement.

The first paper in this area belongs to Melan [1]. He gave the exact solution of the problem for a half-plane (or a whole plane) reinforced by an infinitely long bar to which a concentrated force is applied along its axis. Subsequently Buell [2], Koiter [3] and Brown [4] examined the problem of determination of contact stresses acting between a plate and a semi-infinite bar for different loads applied to the end of this bar.

Reissner was the first to examine the problem of determination of contact stresses in a half-plane with elastic reinforcement of finite length, and he reduced the solution of problem to an integro-differential equation analogous to the Prandtl equation in the theory of a wing with finite span. However, he did not present a solution of this problem. Pflueger obtained the solution of this problem for the case when the upper surface of the elastic cover plate which is joined to the semi-infinite plate is described by an elliptic arc.

Later Benscoter [6] studied the field of stresses in a plate with an elastic reinforcement of finite length for the case where two concentrated forces of equal or opposite direction are applied to its ends. He solved the initial integro-differential equation of the problem by the numerical method of Multhopp.

The work of Kalandiia [7] is devoted to the proof of convergence of the method of Multhopp in its application to an equation of the Prandtl type in the theory of a wing with finite span.

Bufler [8] studied in detail the state of stress in a half-plane (or a whole plane), over a finite section of the free surface (or, correspondingly the inside) of which an elastic cover plate of uniform thickness was attached, under various forms of loading and temperature of influence. For solving of the initial integro-differential equation, which served for the determination of contact stresses between the elastic cover plate and the half-plane, he applied the method used by Carafoli [9 and 10] in the theory of a wing with finite span.

It should be noted here that in all cases mentioned above where the elastic cover plate has a finite length, the obtained solutions are approximate and in addition, these solutions do not always present the possibility to elucidate clearly those peculiarities which characterize the state of stress of the elastic reinforcement in the vicinity of its ends.

In this paper the contact problem is examined for a half-plane with elastic reinforcement of finite length and uniform thickness. The solution of this problem is reduced to an integro-differential equation of the Prandtl type which allows to determine contact stresses along the line of attachement of the elastic cover plate to the half-plane. The exact solution of this equation is presented showing clearly those peculiarities which characterize the state of stress in the vicinity of the ends of the elastic cover plate.

1. Formulation of the problem. The basic equation and its solution. On a finite section $[-a,\ a]$ of its free surface let the half-plane be reinforced by an elastic reincorcement in the form of a welded (or glued) cover plate of constant thickness h (Fig. 1). Let us determine the magnitude and the law of distribution of contact stresses along the line of attachment of the elastic cover plate to the half-plane for the case when a concentrated force P directed along the axis of the cover plate is applied to one of the ends of the cover plate. Let us assume that the bending stiffness of the cover plate is negligibly small and therefore we can neglect the pressure of the cover plate on the half-plane, i.e. to assume that $\sigma_y^{(1)} \approx 0$.

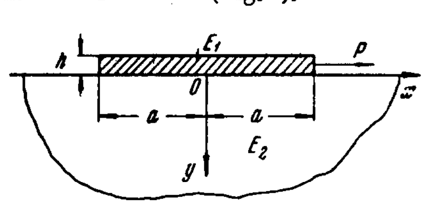

Fig. 1

In other words we shall assume that the cover plate is in a uniaxial state of stress. We shall designate stresses, displacements and deformations in the cover plate by the superscript (1) and in the half-plane by the superscript (2). The physical constants of materials of the cover plate and the half-plane will also be designated in an analogous manner.

From the equilibrium equation for the element of the cover plate we have

$$\sigma_x^{(1)}(x) = \frac{1}{h} \int_{-a}^{x} \tau^{(1)}(s)\, ds \tag{1.1}$$

Here $\sigma_x^{(1)}$ is the normal stress acting in an arbitrary cross section of the cover plate and $\tau^{(1)}(x)$ is the tangential stress acting on the cover plate along the line of its attachment with the half-plane.

Further, taking into account that $\sigma_y^{(1)} \approx 0$, we have in accordance with Hooke's law

$$\varepsilon_x^{(1)} = \frac{du^{(1)}}{dx} = \frac{1}{hE_1} \int_{-a}^{x} \tau^{(1)}(s)\, ds \tag{1.2}$$

Here E_1 is the modulus of elasticity of the cover plate material, h is its thickness and $u^{(1)}$ is the displacement along the x-axis of points of the plane of junction of the cover plate with the elastic half-plane.

On the other hand it is known [11] that the strain $\varepsilon_x^{(2)}$ of the elastic half-plane is expressed through the following formula when tangential forces $\tau^{(2)}(x)$ are acting on a finite section $[-a,\ +a]$ of the free surface

$$\varepsilon_x^{(2)} = \frac{du^{(2)}}{dx} = \frac{2(1-\nu^2)}{\pi E_2} \int_{-a}^{+a} \tau^{(2)}(s) \frac{ds}{s-x} \tag{1.3}$$

Here $u^{(2)}$ is the displacement of points of the free surface of the elastic half-plane along the x-axis, $\tau^{(2)}(s)$ is the tangential stress acting on the elastic half-plane along the line of its contact with the cover plate; E_2 is the modulus of elasticity of the material of the half-plane; ν is Poisson's ratio.

The condition of complete contact of the elastic cover plate with the half-plane in this case can be presented in one of two forms:

$$\text{(A)} \quad u^{(1)} = u^{(2)}, \qquad \text{(B)} \quad \frac{du^{(1)}}{dx} = \frac{du^{(2)}}{dx} \tag{1.4}$$

for $y = 0$ and $-a \leqslant x \leqslant a$, i.e. on the line of contact of the elastic cover plate with the half-plane.

We note that the conditions (A) and (B) differ from one another only by a constant of integration which expresses rigid displacement of the half-plane in the direction of the x-axis and does not have an effect on the state of stress condition of the half-plane.

Utilizing relationships (1.2) and (1.3) and condition (B) we obtain

$$\lambda\varphi(x) + \int_{-a}^{+a} \varphi'(x) \frac{ds}{s-x} = 0 \tag{1.5}$$

Here

$$\varphi'(x) = \tau(x), \quad \varphi(x) = \int_{-a}^{x} \tau(s)\,ds, \quad \tau(x) = \tau^{(1)}(x) = -\tau^{(2)}(x)$$

Here $\tau(x)$ is the contact stress acting between the elastic cover plate and the half-plane, and

$$\lambda = \frac{\pi E_2}{2(1-\nu^2)E_1 h} \tag{1.6}$$

In this manner the solution of the contact problem for the half-plane with the elastic reinforcement of finite length is reduced to the solution of the integro-differential equation (1.5) for boundary conditions

$$\varphi(-a) = 0, \qquad \varphi(a) = -P \tag{1.7}$$

Let us proceed to the solution of the integro-differential equation (1.5). First of all we transform the equation to the form

$$\lambda^*\varphi(x) = \frac{1}{\pi i}\int_{-a}^{+a} \varphi'(s)\frac{ds}{s-x} \qquad \left(\lambda^* = -\frac{\lambda}{\pi i},\ \lambda = \frac{\pi E_2}{2(1-\nu^2)E_1 h}\right) \tag{1.8}$$

Here the integral in the right side should be understood in the sense of the Cauchy principal value.

Using the transformation formula [12] we shall have

$$\varphi'(x) = \tau(x) = \frac{\lambda^*}{\sqrt{a^2-x^2}}\frac{1}{\pi i}\int_{-a}^{+a} \frac{\sqrt{a^2-s^2}\,\varphi(s)ds}{s-x} + \frac{C_1}{\sqrt{a^2-x^2}} \tag{1.9}$$

In this equation that branch of radical $\sqrt{a^2-z^2}$ is taken which acquires positive values on the upper side of section $(-a, +a)$ and C_1 is some constant which is subject to determination.

The solution of the integro-differential equation (1.9) will be sought (assuming that this is possible) in the form of a series

$$\varphi(x) = a_0 + \sum_{k=1}^{k=\infty} a_k \left(\frac{x}{a}\right)^k \tag{1.10}$$

After substitution of (1.10) into (1.9) we obtain

$$\tau(x) = \frac{C_1}{\sqrt{a^2 - x^2}} + \frac{\lambda^*}{\sqrt{a^2 - x^2}} \frac{a_0}{\pi i} \int_{-a}^{+a} \frac{\sqrt{a^2 - s^2}\, ds}{s - x} + \frac{\lambda^*}{\sqrt{a^2 - x^2}} \sum_{k=1}^{k=\infty} a_k J_k(x) \qquad (1.11)$$

where the integral

$$J_k(x) = \frac{1}{\pi i} \int_{-a}^{+a} \left(\frac{s}{a}\right)^k \frac{\sqrt{a^2 - s^2}}{s - x}\, ds \qquad (1.12)$$

should be taken in the sense of the Cauchy principal value.

Applying the formula of Sakhotskii and Plemel [12] we write the integral $J_k(x)$ in the form

$$J_k(x) = -\sqrt{a^2 - x^2}\left(\frac{x}{a}\right)^k + \frac{1}{\pi i} \int_{-a}^{+a} \left(\frac{s}{a}\right)^k \frac{\sqrt{a^2 - s^2}}{s - z}\, ds \qquad (z \to x) \quad (1.13)$$

Here $z = x + i\varepsilon$ where $\varepsilon \to 0$, always remaining positive.

Let us further examine the integral taken over a contour Γ (Fig. 2):

$$I_k(z) = \frac{1}{\pi i} \int_{\Gamma} \left(\frac{s}{a}\right)^k \frac{\sqrt{a^2 - s^2}}{s - z}\, ds \qquad (\Gamma = \Gamma_a + \Gamma_R,\ \Gamma_a = \gamma_a^{(1)} + \gamma_a^{(2)}) \quad (1.14)$$

Let us designate the integral taken over contour Γ_a by $P_k(z)$, i.e.

$$P_k(z) = \frac{1}{\pi i} \int_{\Gamma_a} \left(\frac{s}{a}\right)^k \frac{\sqrt{a^2 - s^2}}{s - z}\, ds \qquad (1.15)$$

Then according to Cauchy's formula we shall have

$$P_k(z) + \frac{1}{\pi i} \int_{\Gamma_R} \left(\frac{s}{a}\right)^k \frac{\sqrt{a^2 - s^2}}{s - z}\, ds = 2\sqrt{a^2 - z^2}\left(\frac{z}{a}\right)^k \qquad (1.16)$$

Now let us compute the integral over the contour Γ_R ($|s| > a$), entering into relationship (1.16).

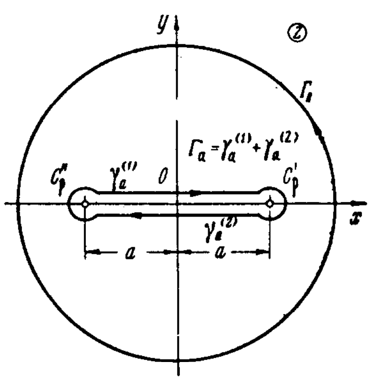

Fig. 2

First of all let us note that the radical in the numerator of the function under the integral acquires on both sides of the section $(-a, +a)$ values which differ only in sign in the geometrically corresponding points; in addition it should be kept in mind that for $s > a$ the radical $\sqrt{a^2 - s^2} = -i\sqrt{s^2 - a^2}$, where $\sqrt{s^2 - a^2} > 0$ for $s > a$.

Now let us use the expansion

$$\left(1 - \frac{a^2}{s^2}\right)^{1/2} = \sum_{\nu=0}^{\nu=\infty} (-1)^\nu C_{1/2}^{(\nu)} \left(\frac{a}{s}\right)^{2\nu} \qquad (1.17)$$

Here $C_{1/2}^{(\nu)}$ is the coefficient of expansion for $|a/s| < 1$.

Then we can give to the integral over Γ_R which enters into relationship (1.16) the form

$$\frac{1}{\pi i} \int_{\Gamma_R} \left(\frac{s}{a}\right)^k \frac{\sqrt{a^2 - s^2}}{s - z}\, ds = -\frac{a}{\pi} \int_{\Gamma_R} \sum_{\nu=0}^{\nu=\infty} (-1)^\nu C_{1/2}^{(\nu)} \left(\frac{a}{s}\right)^{2\nu} \left(\frac{s}{a}\right)^{k+1} \frac{ds}{s - z} \qquad (1.18)$$

Let us examine the sum which is under the integral sign in the right side of (1.18) and let us designate it through V

$$V = \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} C_{1/2}^{(\nu)} \left(\frac{s}{a}\right)^{k+1-2\nu} \tag{1.19}$$

It is apparent that positive powers $(s\,/\,a)$ in the sum (1.19) will occur for

$$k+1-2\nu \geqslant 0, \qquad \nu \leqslant \nu_{*}, \qquad \nu_{*} = \mathrm{E}\,[{}^{1}/_{2}\,(k+1)]$$

Here $\mathrm{E}\,[\,{}^{1}/_{2}\,(k+1)]$ is an integral part of ${}^{1}/_{2}\,(k+1)$. In this manner we shall have

$$V = \sum_{\nu=0}^{\nu=\nu_{*}} (-1)^{\nu} C_{1/2}^{(\nu)} \left(\frac{s}{a}\right)^{k+1-2\nu} + \sum_{\nu=\nu_{*}+1}^{\nu=\infty} (-1)^{\nu} C_{1/2}^{(\nu)} \left(\frac{s}{a}\right)^{k+1-2\nu} \tag{1.20}$$

In the first sum of this relationship we repalce the index ν by $\mu = k+1-2\nu$; then we will have

$$\mu = k+1 \quad \text{for } \nu = 0, \qquad \mu_{*} = k+1-2\mathrm{E}\,[{}^{1}/_{2}\,(k+1)] \quad \text{for } \nu = \nu_{*} \tag{1.21}$$

It is apparent that if k is even, then $\mu_{*} = 1$, and if k is uneven, $\mu_{*} = 0$. Further, keeping in mind that $\nu = {}^{1}/_{2}\,(k+1-\mu)$, the first term of the relationship (1.20) can be presented in the form

$$\sum_{\nu=0}^{\nu_{*}} (-1)^{\nu} C_{1/2}^{(\nu)} \left(\frac{s}{a}\right)^{k+1-2\nu} = \sum_{\mu=0;\,1}^{k+1}{}^{*} (-1)^{{}^{1}/_{2}(k+1-\mu)} C_{1/2}^{({}^{1}/_{2}[k+1-\mu])} \left(\frac{s}{a}\right)^{\mu} \tag{1.22}$$

where the asterisk above the symbol of summation indicates that the index of summation takes either even or uneven values. Let us now examine the second term in relationship (1.20)

$$\sum_{\nu=\nu_{*}+1}^{\infty} (-1)^{\nu} C_{1/2}^{(\nu)} \left(\frac{s}{a}\right)^{k+1-2\nu} = \sum_{\nu=\mathrm{E}\,[{}^{1}/_{2}(k+1)]+1}^{\infty} (-1)^{\nu} C_{1/2}^{(\nu)} \left(\frac{a}{s}\right)^{2\nu-(k+1)}$$

Here we also introduce a new index of summation $\mu = 2\nu-(k+1)$; then we will have

$$\mu = \infty \qquad \text{for } \nu = \infty$$

$$\mu_{*} = 2\mathrm{E}\,[{}^{1}/_{2}\,(k+1)]-(k-1) \qquad \text{for } \nu = \nu_{*}+1 = \mathrm{E}\,[{}^{1}/_{2}\,(k+1)]+1 \tag{1.23}$$

It is apparent that if k is even, then $\mu_{*} = 1$, and if k is uneven then $\mu_{*} = 2$. We note that $\nu = {}^{1}/_{2}\,(\mu+k+1)$; then the second term of relationship (1.20) can be presented in the form

$$\sum_{\nu=\nu_{*}+1}^{\infty} (-1)^{\nu} C_{1/2}^{(\nu)} \left(\frac{s}{a}\right)^{k+1-2\nu} = \sum_{\mu=2;\,1}^{\infty}{}^{*} (-1)^{{}^{1}/_{2}(k+1+\mu)} C_{1/2}^{({}^{1}/_{2}[k+1+\mu])} \left(\frac{a}{s}\right)^{\mu} \tag{1.24}$$

Now the numerator of the function under the integral in (1.18), i.e. expression (1.20) can be represented in the form

$$V = \sum_{\mu=0;\,1}^{k+1}{}^{*} (-1)^{{}^{1}/_{2}(k+1-\mu)} C_{1/2}^{({}^{1}/_{2}[k+1-\mu])} \left(\frac{s}{a}\right)^{\mu} +$$

$$+ \sum_{\mu=2;\,1}^{\infty}{}^{*} (-1)^{{}^{1}/_{2}(k+1+\mu)} C_{1/2}^{({}^{1}/_{2}[k+1+\mu])} \left(\frac{a}{\ }\right)^{\mu} \tag{1.25}$$

Substituting Expression (1.25) into the right side of integral (1.18) we obtain

$$\frac{1}{\pi i}\int_{\Gamma_R}\left(\frac{s}{a}\right)^k\frac{\sqrt{a^2-s^2}}{s-z}\,ds=-\frac{a}{\pi}\int_{\Gamma_R}\frac{ds}{s-z}\left\{\sum_{\mu=0;\,1}^{k+1}{}^{*}(-1)^{1/2(k+1-\mu)}\times\right.$$

$$\left.\times\, C_{1/2}^{(1/2[k+1-\mu])}\left(\frac{s}{a}\right)^{\mu}+\sum_{\mu=2;\,1}^{\infty}{}^{*}(-1)^{1/2(k+1+\mu)}C_{1/2}^{(1/2[k+1+\mu])}\left(\frac{a}{s}\right)^{\mu}\right\} \quad (1.26)$$

On the basis of the residue theorem we have

$$\int_{\Gamma_R}\frac{ds}{s-z}\left(\frac{s}{a}\right)^{\mu}=2\pi i\left(\frac{z}{a}\right)^{\mu} \qquad (\mu\geqslant 0) \quad (1.27)$$

$$\int_{\Gamma_R}\frac{ds}{s-z}\left(\frac{a}{s}\right)^{\mu}=0 \qquad (\mu\geqslant 1) \quad (1.28)$$

Substituting values of these integrals into relationship (1.26) we find

$$\frac{1}{\pi i}\int_{\Gamma_R}\left(\frac{s}{a}\right)^k\frac{\sqrt{a^2-s^2}}{s-z}\,ds=-2ia\sum_{\mu=0;\,1}^{k+1}{}^{*}(-1)^{1/2(k+1-\mu)}C_{1/2}^{(1/2[k+1-\mu])}\left(\frac{z}{a}\right)^{\mu} \quad (1.29)$$

Then, using relationship (1.16) and (1.29) we obtain for $P_k(z)$ the following expressions:

$$P_k(z)=2\sqrt{a^2-z^2}\left(\frac{z}{a}\right)^k+2ia\sum_{\mu=0;\,1}^{k+1}{}^{*}(-1)^{1/2(k+1-\mu)}C_{1/2}^{(1/2[k+1-\mu])}\left(\frac{z}{a}\right)^{\mu} \quad (1.30)$$

But on the other hand according to (1.15) we have

$$P_k(z)=\frac{1}{\pi i}\int_{\gamma_a^{(1)}}\left(\frac{s}{a}\right)^k\frac{\sqrt{a^2-s^2}}{s-z}\,ds+\frac{1}{\pi i}\int_{\gamma_a^{(2)}}\left(\frac{s}{a}\right)^k\frac{\sqrt{a^2-s^2}}{s-z}\,ds \quad (1.31)$$

Here the first integral is taken along the upper side of section $\gamma_a^{(1)}$ and the second integral along the lower side of the section $\gamma_a^{(2)}$ (Fig. 2)(it is apparent that integrals with respect to small regions C_ρ' and C_ρ'' tend to zero for $\rho\to 0$), consequently

$$P_k(z)=\frac{1}{\pi i}\int_{-a}^{+a}\left(\frac{s}{a}\right)^k\frac{\sqrt{a^2-s^2}}{s-z}\,ds-\frac{1}{\pi i}\int_{+a}^{-a}\left(\frac{s}{a}\right)^k\frac{\sqrt{a^2-s^2}}{s-z}\,ds \quad (1.32)$$

Here that branch radical $\sqrt{a^2-s^2}$, is selected which remains positive on the upper side of the section $(-a,\ a)$ and by the same token takes on negative values on the lower side.

In this manner we have according to (1.32)

$$P_k(z)=\frac{2}{\pi i}\int_{-a}^{+a}\left(\frac{s}{a}\right)^k\frac{\sqrt{a^2-s^2}}{s-z}\,ds \quad (1.33)$$

Now using relationship (1.33) we can present Expression (1.13) for $J_k(x)$ in the form

$$J_k(x)=-\sqrt{a^2-x^2}\left(\frac{x}{a}\right)^k+\frac{1}{2}P_k(x) \quad (1.34)$$

Here $P_k(x)$ is understood to mean the limiting value on the upper side of the section $(-a,+a)$ of the function $P_k(z)$ determined by the relationship (1.30) or (1.33).

Substituting the value $P_k(x)$ determined by the relationship (1.30) into (1.34) and cancelling $\sqrt{a^2-x^2}\,(x/a)^k$, we obtain the following Formula:

$$J_k(x) = \frac{1}{\pi i}\int_{-a}^{+a}\left(\frac{s}{a}\right)^k \frac{\sqrt{a^2-s^2}}{s-x}\,ds = ia \sum_{\mu=0;\,1}^{k+1}{}^{*} (-1)^{1/2(k+1-\mu)}\, C_{1/2}^{(1/2[k+1-\mu])}\left(\frac{x}{a}\right)^{\mu} \tag{1.35}$$

Now let us turn to Expression (1.11) for the contact stress $\tau(x)$. First of all we note that the integral entering into this equation for a_0 is

$$J_0(x) = \frac{1}{\pi i}\int_{-a}^{+a}\frac{\sqrt{a^2-s^2}\,ds}{s-x} = ix \tag{1.36}$$

It is obtained directly from Eq. (1.35) for $k = 0$, $\mu = 1$ and $C_{1/2}^{(0)} = 1$.

Substituting values of integrals $J_k(x)$ and $J_0(x)$ determined from relationships (1.35) and (1.36) into Eq. (1.11) for the contact stress $\tau(x)$, we obtain

$$\tau(x) = \frac{ia\lambda^*}{\sqrt{a^2-x^2}}\sum_{k=1}^{k=\infty} a_k \sum_{\mu=0;\,1}^{k+1}{}^{*} (-1)^{1/2(k+1-\mu)}\, C_{1/2}^{(1/2[k+1-\mu])}\left(\frac{x}{a}\right)^{\mu} + \frac{C_1 + i\lambda^* a_0 x}{\sqrt{a^2-x^2}} \tag{1.37}$$

We introduce the notation

$$\lambda^* = -\frac{\lambda_0}{i}, \qquad \lambda_0 = \frac{E_2}{2(1-\nu^2)E_1 h}$$

Then Formula (1.37) for contact stress can be written in the form

$$\tau(x) = -\frac{\lambda_0 a}{\sqrt{a^2-x^2}}\sum_{k=1}^{k=\infty} a_k \sum_{\mu=0;\,1}^{k+1}{}^{*} (-1)^{1/2(k+1-\mu)}\, C_{1/2}^{(1/2[k+1-\mu])}\left(\frac{x}{a}\right)^{\mu} + \frac{C_1 - \lambda_0 a_0 x}{\sqrt{a^2-x^2}} \tag{1.38}$$

Admitting that a transposition is possible in the double sum in Eq. (1.38), after some transformations we bring the latter to the form

$$\tau(x) = -\frac{\lambda_0 a}{\sqrt{a^2-x^2}}\sum_{\mu=0}^{\infty}\left(\frac{x}{a}\right)^{\mu}\sum_{k=N(\mu)}^{\infty}{}^{*} a_k (-1)^{1/2(k+1-\mu)}\, C_{1/2}^{(1/2[k+1-\mu])} + \frac{C_1 - \lambda_0 a_0 x}{\sqrt{a^2-x^2}} \tag{1.39}$$

Here

$$N(\mu) = \begin{cases} 1 & \text{for } \mu = 0 \\ 2 & \text{for } \mu = 1 \\ \mu - 1 & \text{for } \mu \geqslant 2 \end{cases} \tag{1.40}$$

and the asterisk to the right of the summation sign indicates that the index of the internal sum assumes either even or odd values.

Let us write

$$B_\mu = \sum_{k=N(\mu)}^{\infty}{}^{*} a_k (-1)^{1/2(k+1-\mu)}\, C_{1/2}^{(1/2[k+1-\mu])} \tag{1.41}$$

Then Eq. (1.39) for the contact stress $\tau(x)$ can be presented in the form

$$\tau(x) = -\frac{\lambda_0 a}{\sqrt{a^2-x^2}}\sum_{\mu=0}^{\infty} B_\mu\left(\frac{x}{a}\right)^{\mu} + \frac{C_1 - \lambda_0 a_0 x}{\sqrt{a^2-x^2}} \tag{1.42}$$

As is evident from this equation, those singularities are clearly brought out in it which are inherent to contact stresses in the vicinity of the ends of the elastic cover plate.

In this manner the magnitude and the distribution law of the contact stress $\tau(x)$ acting in the plane of contact of the elastic cover plate with the half-plane, is completely determined either by Eq. (1.38) or (1.39) or (1.42), if the values of the coefficients a_k or B_μ are known.

It is shown below that the determination of coefficients a_k (or B_μ) is reduced to the solution of some infinite system of linear equations with bounded free terms.

Simultaneously it is proved that for

$$\lambda_0 a = \frac{E_2}{2(1-\nu^2)E_1}\frac{a}{h} < 1$$

this infinite system of linear equations turns out to be completely regular and for $\lambda_0 a \geqslant 1$ quasi-completely regular and as is known [13] from the theory of regular infinite systems of linear equations, in the case of bounded free terms the unknowns are determined with any required accuracy.

2. Derivation and analysis of an infinite system of linear equations. First of all we transform Eq. (1.39) for the contact stress $\tau(x)$, representing it in the following form:

$$\tau(x) = -\lambda_0 \sum_{n=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^n C_{-1/2}^{(n)} B_\mu \left(\frac{x}{a}\right)^{2n+\mu} +$$

$$+\left[\frac{C_1}{a} - \lambda_0 a_0\left(\frac{x}{a}\right)\right]\sum_{n=0}^{\infty}(-1)^n C_{-1/2}^{(n)}\left(\frac{x}{a}\right)^{2n} \tag{2.1}$$

This expression is obtained directly from Eq. (1.39), if we substitute in it

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n C_{-1/2}^{(n)}\left(\frac{x}{a}\right)^{2n}, \quad \left|\frac{x}{a}\right| < 1 \tag{2.2}$$

where $C_{-1/2}^{(n)}$ is the coefficient of expansion.

Now we write the first sum entering into Expression (2.1) substituting in it the index of summation $2n+\mu$ by m. Then we shall have

$$I_1 = -\lambda_0 \sum_{\mu=0}^{\infty}\sum_{\mu=m}^{\infty}{}^{*}(-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} B_\mu \left(\frac{x}{a}\right)^m \tag{2.3}$$

Further, changing the order of summation we obtain

$$I_1 = -\lambda_0 \sum_{m=0}^{\infty}\left(\frac{x}{a}\right)^m \sum_{\mu=0;\,1}^{m}{}^{*}(-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} B_\mu \tag{2.4}$$

Substituting the index of summation $2n$ by m in the second sum which enters into Expression (2.1), we reduce it to the form (2.5)

$$I_2 = \frac{C_1}{a}\sum_{m=0}^{\infty}{}^{*}(-1)^{1/2m} C_{-1/2}^{(1/2m)}\left(\frac{x}{a}\right)^m - \lambda_0 a_0 \sum_{m=1}^{\infty}{}^{*}(-1)^{1/2(m-1)} C_{-1/2}^{(1/2[m-1])}\left(\frac{x}{a}\right)^m$$

Now let us represent the contact stress $\tau(x)$ in the form of a sum of two terms: of symmetric $\tau^+(x)$ and skew-symmetric $\tau^-(x)$ stresses, i.e. $\tau(x) = \tau^+(x) + \tau^-(x)$. It is apparent that the symmetrical part of the contact stress $\tau^+(x)$ will depend only on even powers of $(x/a)^m$, and the skew-symmetric part only on odd powers $(x/a)^m$.

Thus for even m we shall have

$$\tau^+(x) = -\lambda_0 \sum_{m=0}^{\infty}{}^{*}\left(\frac{x}{a}\right)^m\left\{\sum_{\mu=0}^{\mu=m}{}^{*}(-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} B_\mu -\right.$$

$$\left. - \frac{C_1}{\lambda_0 a}(-1)^{1/2m} C_{-1/2}^{(1/2m)}\right\} \tag{2.6}$$

for odd m

$$\tau^-(x) = -\lambda_0 \sum_{m=1}^{\infty}{}^{*} \left(\frac{x}{a}\right)^m \left\{ \sum_{\mu=1}^{\mu=m}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} B_\mu + \right.$$

$$\left. + a_0 (-1)^{1/2(m-1)} C_{-1/2}^{(1/2[m-1])} \right\} \tag{2.7}$$

On the other hand according to (1.10) we have

$$\varphi(x) = a_0 + \sum_{k=1}^{\infty} a_k \left(\frac{x}{a}\right)^k \tag{2.8}$$

Consequently

$$\tau(x) = \varphi'(x) = \sum_{m=0}^{\infty} \frac{a_{m+1}(m+1)}{a} \left(\frac{x}{a}\right)^m, \qquad \left|\frac{x}{a}\right| < 1 \tag{2.9}$$

In this case the singularity at the ends of the elastic cover plate $(x = \mp a)$ is taken care of either by Eq. (1.38) or (1.39) or (1.42).

Let us separate the even and odd parts in Expression (2.9); for even m we obtain

$$\tau^+(x) = \sum_{m=0}^{\infty}{}^{*} \frac{a_{m+1}(m+1)}{a} \left(\frac{x}{a}\right)^m \tag{2.10}$$

and for odd m we shall have

$$\tau^-(x) = \sum_{m=1}^{\infty}{}^{*} \frac{a_{m+1}(m+1)}{a} \left(\frac{x}{a}\right)^m \tag{2.11}$$

Further, comparing Expressions (2.6) and (2.10) and also (2.7) and (2.11) we find

$$a_{m+1} \frac{m+1}{a} = -\lambda_0 \left\{ \sum_{\mu=0}^{m}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} B_\mu - \frac{C_1}{\lambda_0 a} (-1)^{1/2 m} C_{-1/2}^{(1/2 m)} \right\} \tag{2.12}$$

$$a_{m+1} \frac{m+1}{a} = -\lambda_0 \left\{ \sum_{\mu=1}^{m}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[(m-\mu])} B_\mu + a_0 (-1)^{1/2(m-1)} C_{-1/2}^{(1/2[m-1)]} \right\} \tag{2.13}$$

In Eq. (2.12) m is even and in (2.13) it is odd.

In this manner the system of equations (2.12) contains only coefficients a_k with odd index k, which linearly depend on an arbitrary constant C_1 and determine according to Eq. (2.11) the contact stress $\tau^-(x)$ in the case of skew-symmetric loading of the elastic cover plate. The system of equations (2.13) contains only coefficients a_k with an even index k which also depend linearly on an arbitrary constant a_0 and determine according to Eq. (2.10) the contact stress $\tau^+(x)$ in the case of symmetric loading of the elastic cover plate. These arbitrary constants C_1 and a_0 are determined from boundary conditions (1.7).

It is appropriate to note here that each of these loadings is of independent interest because it corresponds to a definite character of deformation of the half-plane with a cover plate.

In Eqs. (2.12) and (2.13) let us now transform sums which contain coefficients B_μ using in this connection relationship (1.41). Designating this sum by A_m, we shall have

$$A_m = \sum_{\mu=0;\,1}^{m}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} B_\mu \tag{2.14}$$

Substituting expression for B_μ from relationship (1.41) into (2.14), we obtain

$$A_m = \sum_{\mu=0}^{m} \varepsilon_{\mu,m} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} \sum_{k=N(\mu)}^{\infty} \varepsilon_{\mu+1,k} a_k (-1)^{1/2(k+1-\mu)} C_{1/2}^{(1/2[k+1-\mu])} \tag{2.15}$$

Here

$$\varepsilon_{\mu,m} = \begin{cases} 1 & \text{for } \mu \text{ and } m \text{ with the same parity} \\ 0 & \text{for } \mu \text{ and } m \text{ with different parity} \end{cases} \tag{2.16}$$

$$N(\mu) = \begin{cases} 1 & \text{for } \mu = 0 \\ 2 & \text{for } \mu = 1 \\ \mu - 1 & \text{for } \mu \geqslant 2 \end{cases} \tag{2.17}$$

Changing the order of summation in (2.15) we can obtain

$$A_m = \sum_{k=1}^{m-1} \sum_{\mu=0}^{k+1} f_{\mu k}(m) + \sum_{k=m}^{\infty} \sum_{\mu=0}^{m} f_{\mu k}(m) \tag{2.18}$$

where expressions for $f_{\mu k}(m)$ are determined by Eqs.

$$f_{\mu k}(m) = a_k \varepsilon_{\mu,m} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} \varepsilon_{\mu+1,k} (-1)^{1/2(k+1-\mu)} C_{1/2}^{(1/2[k+1-\mu])} \tag{2.19}$$

Let us now separate in Expression (2.18) for the sum A_m its even and odd parts. The even part will be denoted by A_m^+ and the odd part through A_m^-.

Further, let us introduce the following notations:

$$g_{k,m} = \sum_{\mu=0;1}^{\min(k,m)}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} (-1)^{1/2(k-\mu)} C_{1/2}^{(1/2[k-\mu])} \tag{2.20}$$

Then by virtue of relationships (2.18), (2.19) and (2.20) we obtain

$$A_m^+ = \sum_{k=1}^{m-1}{}^{*} a_k \sum_{\mu=0}^{k+1}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} (-1)^{1/2(k+1-\mu)} C_{1/2}^{(1/2[k+1-\mu])} + \tag{2.21}$$

$$+ \sum_{k=m+1}^{\infty} a_k \sum_{\mu=0}^{m}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} (-1)^{1/2(k+1-\mu)} C_{1/2}^{(1/2[k+1-\mu])} = \sum_{k=1}^{\infty}{}^{*} a_k g_{k+1,m}$$

Here m is even, k is odd and $g_{k+1,m}$ is determined from Eq. (2.20). Further,

$$A_m^- = \sum_{k=2}^{m-1}{}^{*} a_k \sum_{\mu=1}^{k+1}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} (-1)^{1/2(k+1-\mu)} C_{1/2}^{(1/2[k+1-\mu])} + \tag{2.22}$$

$$+ \sum_{k=m+1}^{\infty}{}^{*} a_k \sum_{\mu=1}^{m}{}^{*} (-1)^{1/2(m-\mu)} C_{-1/2}^{(1/2[m-\mu])} (-1)^{1/2(k+1-\mu)} C_{1/2}^{(1/2[k+1-\mu])} = \sum_{k=2}^{\infty}{}^{*} a_k g_{k+1,m}$$

Here m is odd, k is even, and $g_{k+1,m}$ is also determined from Eq. (2.20).

Now substituting according to (2.14) values of A_m^+ and A_m^- from relationships (2.21) and (2.22) into Eqs. (2.12) and (2.13) we reduce then to the following form:

$$a_{m+1} \frac{m+1}{a} = -\lambda_0 \left\{ \sum_{k=1}^{\infty}{}^{*} a_k g_{k+1,m} - \frac{C_1}{a_0 \lambda_0} (-1)^{1/2 m} C_{-1/2}^{(1/2 m)} \right\} \tag{2.23}$$

$$a_{m+1} \frac{m+1}{a} = -\lambda_0 \left\{ \sum_{k=2}^{\infty}{}^{*} a_k g_{k+1,m} + a_0 (-1)^{1/2(m-1)} C_{-1/2}^{(1/2[m-1])} \right\} \tag{2.24}$$

In Eq. (2.23) m is even $(m = 0, 2, 4, 6, \ldots)$ and k is odd and in Eq. (2.24) m is odd $(m = 1, 3, 5, 7, \ldots)$ and k is even.

In this manner the problem of determining unknown coefficients a_k $(k = 1, 2, 3, \ldots)$ (or B_k), entering into Eqs. (1.38), (1.39) or (1.42) for determination of contact stress $\tau(x)$ was reduced to the solution of an infinite system of linear algebraic equations (2.23) and (2.24).

In this connection, coefficients a_k with odd index k, which are expressed through an arbitrary constant C_1, are determined from the system of equations (2.23). From the system of equations (2.24) coefficients a_k with even index k, which depend on the arbitrary constant a_0, are determined. Both of these arbitrary constants C_1 and a_0 enter separately in the form of a factor into the corresponding expressions for a_k with odd indices and are determined from boundary conditions (1.7).

Further, following the method developed by Sherman in [14] we shall prove that the infinite system of linear equations (2.23) and (2.24) is completely regular if $\lambda_0 a < 1$ and quasi-completely regular when $\lambda_0 a \geqslant 1$.

For this purpose we first of all transform our system into a form which is convenient for analysis, that is we write it in the form

$$a_{m+1}\left[\frac{m+1}{a} + \lambda_0 g_{m+2,m}\right] = -\lambda_0\left\{\sum_{k=1}^{\infty}{}^{*} a_k g_{k+1,m} - \frac{C_1}{a\lambda_0}(-1)^{\frac{1}{2}m} C_{-\frac{1}{2}}^{(\frac{1}{2}m)}\right\} \tag{2.25}$$

where $m = 0, 2, 4, 6, \ldots,\ k \neq m+1$ and

$$a_{m+1}\left[\frac{m+1}{a} + \lambda_0 g_{m+2,m}\right] = -\lambda_0\left\{\sum_{k=2}^{\infty}{}^{*} a_k g_{k+1,m} + a_0(-1)^{\frac{1}{2}(m-1)} C_{-\frac{1}{2}}^{(\frac{1}{2}[m-1])}\right\} \tag{2.26}$$

where $m = 1, 3, 5, 7, \ldots$ and $k \neq m+1$.

In order to prove that these systems sre completely regular it is necessary for us to evaluate the following sums:

$$\sum_{k=1}^{\infty}{}^{*} |g_{k+1,m}| \quad \text{for} \quad k \neq m+1;\ m = 0, 2, 4, 6, \ldots \tag{2.27}$$

$$\sum_{k=2}^{\infty}{}^{*} |g_{k+1,m}| \quad \text{for} \quad k \neq m+1;\ m = 1, 3, 5, 7, \ldots \tag{2.28}$$

For this purpose we shall take advantage of some relationships obtained in paper [14]. For distinction these relationships are designated by a serial number with an asterisk. According to [14] the following relationships apply:

$$g_{k,m} = g_{k-2,m-2} + g_{1,m} g_{k,1}, \qquad g_{k,m} = g_{k-1,m-1} \tag{2.29*}$$

for odd k and m. In addition to this we introduce for even indices k and m Eq.

$$g_{2k,2m} = \frac{2m+1}{2(m-k)+1}(-1)^k C_{-\frac{1}{2}}^{(k)} (-1)^m C_{-\frac{1}{2}}^{(m)} \tag{2.30*}$$

and also the following expressions for sums:

$$\sum_{k=2}^{\infty}{}^{*} |g_{k,m}| = -(-1)^{\frac{1}{2}m} C_{-\frac{1}{2}}^{(\frac{1}{2}m)} + 2\sum_{\nu=0}^{\frac{1}{2}m} [C_{-\frac{1}{2}}^{(\nu)}]^2 \tag{2.31*}$$

$$\sum_{k=1}^{\infty}{}^{*} |g_{k,m}| = 2\sum_{\nu=0}^{\frac{1}{2}(m-1)} [C_{-\frac{1}{2}}^{(\nu)}]^2 \tag{2.32*}$$

As before, an asterisk to the right of the summation denotes that the index of summation takes on either even or odd values. It is necessary to keep in mind that $g_{m+2,m} \neq 0$

From the obvious equation $(C_{-1/2}^{(0)} = 1,\ C_{-1/2}^{(1)} = -1/2)$

$$(-1)^n C_{-1/2}^{(n)} = \frac{1}{n!}\frac{1}{2}\left(1+\frac{1}{2}\right)\cdots\left(n-1+\frac{1}{2}\right) > 0 \quad \text{for } n \geqslant 1 \qquad (2.33^*)$$

the following inequalities follow directly

$$|C_{-1/2}^{(n)}| < |C_{-1/2}^{(n-1)}| \quad (n \geqslant 1) \qquad (2.34)$$

$$\text{(a)}\quad |C_{-1/2}^{(n)}| \leqslant \frac{1}{2} \quad (n \geqslant 1), \qquad \text{(b)}\quad |C_{-1/2}^{(n)}| > \frac{1}{2n} \quad (n \geqslant 1) \qquad (2.35)$$

Relationships (2.29*) – (2.33*) and inequalities (2.34) and (2.35) will be needed later.

Let us proceed to the analysis of infinite systems of linear algebraic equations (2.25) and (2.26).

It is known [13] that for the system of infinite equations (2.25) (or (2.26)) to be completely regular it is necessary that

$$M_m = \frac{\lambda_0}{|(m+1)/a + \lambda_0 g_{m+2,m}|} \sum_{k=1}^{\infty}{}^{*} |g_{k+1,m}| < 1 \qquad (2.36)$$

for all values of $m = 0, 2, 4, 6, \ldots$ and $k \neq m+1$. In particular, it must be that $M_0 < 1$.

Changing in Eq. (2.36) the index of summation $k+1$ to n and using relationship (2.31*), we obtain

$$M_m = \frac{1}{|(m+1)/a\lambda_0 + g_{m+2,m}|}\left\{\sum_{n=2}^{\infty}{}^{*} |g_{n,m}| - |g_{m+2,m}|\right\} = \qquad (2.37)$$

$$= \frac{1}{|(m+1)/a\lambda_0 + g_{m+2,m}|}\left\{-(-1)^{1/2 m} C_{-1/2}^{(1/2 m)} + 2\sum_{\nu=0}^{1/2 m} [C_{-1/2}^{(\nu)}]^2 - |g_{m+2,m}|\right\} < 1$$

for all values $m = 0, 2, 4, 6, \ldots$

From Eq. (2.37) it follows immediately that

$$M_0 = \left|\frac{a\lambda_0}{2 - a\lambda_0}\right| < 1 \qquad (2.38)$$

In this manner in the following examination of condition (2.36) or (2.37) for the infinite system of equations (2.25) it is meaningful to take a $a\lambda_0 < 1$. Simultaneously we note that the denominator in Expression (2.37), i. e.

$$N_m = \frac{m+1}{a\lambda_0} + g_{m+2,m} \qquad (2.39)$$

is by virtue of inequality (2.34) a monotonically decreasing function of m $(N_m > 0)$ which reaches its minimum value $N_0 = 1/2$ for $m = 0$. In this connection the inequality $g_{m+2,m} < 0$ always holds.

Then the condition (2.37) can be written in the form

$$-(-1)^{1/2 m} C_{-1/2}^{(1/2 m)} + 2\sum_{\nu=0}^{1/2 m} [C_{-1/2}^{(\nu)}]^2 - |g_{m+2,m}| < \frac{m+1}{a\lambda_0} - |g_{m+2,m}| \qquad (2.40)$$

since $N_m > 0$ then, consequently,

$$|N_m| = \frac{m+1}{a\lambda_0} + g_{m+2,m} = \frac{m+1}{a\lambda_0} - |g_{m+2,m}|$$

Now condition (2.40) is written in the form of an inequality placed on the quantity $a\lambda_0$. From (2.40) we have directly

$$a\lambda_0 < H_m = (m+1)\left\{-(-1)^{1/2m} C_{-1/2}^{(1/2m)} + 2\sum_{\nu=0}^{1/2m} [C_{-1/2}^{(\nu)}]^2\right\}^{-1} \quad (2.41)$$

This inequality must hold for $m = 0, 2, 4, 6, \ldots$

In this manner, if the quantity $a\lambda_0$ satisfies inequality (2.41) for all values $m = 0, 2, 4, 6, 8,$ then in this case the infinite system of linear equations (2.25) will be completely regular.

From inequality (2.41) immediately follows the result that $a\lambda_0 < 1$ for $m = 0$, as obtained earlier.

Therefore it remains to be shown whether for $a\lambda_0 < 1$ the condition (2.41) will be preserved for all remaining values of $m = 2, 4, 6, 8, \ldots$, i.e. the condition for which the infinite system of equations (2.25) is completely regular, or whether it will be necessary in this case to impose stronger restictions on the quantity $a\lambda_0$. It is proved below that for $a\lambda_0 < 1$ the condition (2.41) is also satisfied for all other values of $m = 2, 4, 6, 8, \ldots$, i.e. the infinite system of linear equations will be completely regular if only $a\lambda_0 < 1$. In fact, we introduce the notation

$$\Delta_m = -(-1)^{1/2m} C_{-1/2}^{(1/2m)} + 2\sum_{\nu=0}^{1/2m} [C_{-1/2}^{(\nu)}]^2 \quad (2.42)$$

Using inequalities (2.35) we can show that

$$\Delta_m \leqslant \frac{m^2+8m-4}{4m} \quad (2.43)$$

for all values $m = 2, 4, 6, 8, \ldots$ It is apparent that $m^2 + 8m - 4 > 0$ for all values $m = 2, 4, 6, 8, \ldots$

On the other hand we have by virtue of inequalities (2.43) and (2.41)

$$H_m = \frac{m+1}{\Delta_m} \geqslant \frac{4m(m+1)}{m^2+8m-4} = \theta(m) \quad (2.44)$$

for all values $m = 2, 4, 6, 8, \ldots$

Therefore if $a\lambda_0$' will satidfy the inequality

$$a\lambda_0 < \min\theta(m) = \frac{4m(m+1)}{m^2+8m-4} \quad (2.45)$$

the condition (2.41) will be satisfied all the more for values $m = 2, 4, 6, 8, \ldots$

Now let us examine $\theta(m)$ as a function of m. This function has a minimum for $m > 1$. In fact

$$\theta'(m) = 4\frac{7m^2-8m-4}{(m^2+8m-4)^2} = 0 \quad (2.46)$$

has one positive root which is between 1 and 2. Therefore $\min\theta(m)$ in case of discrete integer values of $m = 2, 4, 6, 8, \ldots$, is achieved for $m = 2$ as a result of the monotonic character of function $\theta(m)$, i.e. the following sufficient conditions holds for the infinite system of equations (2.25) to be completely regular:

$$a\lambda_0 < \left.\frac{4m(m+1)}{m^2+8m-4}\right|_{m=2} = \frac{3}{2} \quad (2.47)$$

Thus for $a\lambda_0 < 3/2$ the condition (2.41) is automatically satisfied for values of $m = 2, 4, 6, 8, \ldots$ Consequently the infinite system of linear equations (2.25) will be completely regular for $a\lambda_0 < 1$.

In an analogous manner it is not difficult to prove that for this condition, i.e. for

$a\lambda_0 < 1$ the infinite system of linear equations (2.26) will also be completely regular.

In conclusion we note that from the entire development of the proof presented above for the statement that the infinite system of equations (2.25) or (2.26) is completely regular it follows simultaneously that if $a\lambda_0 > 1$, then these systems will be quasi-completely regular and for each fixed value $a\lambda_0$ (of course, under condition that $a\lambda_0 > 1$) we can always point out that value of m ($m = 2, 4, 6, \ldots$ or $m = 1, 3, 5, \ldots$), starting with which the infinite system of equations will be completely regular. Simulataneously it is directly evident from the structure of Eqs. (2.25) and (2.26) that the free terms of these equations are bounded and tend to zero for $m \to \infty$ as $O(m^{-1})$.

Thus, having constructed for finding coefficients a completely regular (for $a\lambda_0 < 1$) and a quasi-completely regular infinite system of linear equations with bounded free terms, we can find with required accuracy the values of coefficients which enter into Eq. (1.38) or (1.39) or (1.42) for the contact stress $\tau(x)$.

The author is grateful to D. I. Sherman for discussion of this work and for useful comments.

BIBLIOGRAPHY

1. Melan, E., Ein Beitrag zur Theorie geschweisster Verbindungen. Ing.-Archiv, Bd. 3, Heft 2, 1932.
2. Buell, E. L., On the distribution of plane stress in a semi-infinite plate with partially stiffened edge. J. Math. Phys., Vol. 26, 1948.
3. Koiter, W. T., On the diffusion of load from a stiffener into a sheet. Quart. J. Mech. and Appl. Math., Vol. 8, 1955.
4. Brown, E. H., The diffusion of load from a stiffener into an infinite elastic sheet. Proc. Roy. Soc. Ser. A, Vol. 239, №1218, 1957.
5. Reissner, E., Note on the problem of the distribution of stress in a thin stiffened elastic sheet. Proc. Nat. Acad. of Sci. U.S.A., Vol. 26, 1940.
6. Benscoter, E. U., Analysis of single stiffener on an infinite sheet. J. Appl. Mech., Vol. 16, №3, 1949.
7. Kalandiia, A. I., On a direct method of solution of the equation of the wing theory and its application in the theory of elasticity. Matem. sb. №2, 1957.
8. Bufler, H., Scheibe mit endlicher elasticher Versteifung. VDI-Forschungsh. №485, 1961.
9. Bufler, H., Einige strenge Lösungen für den Spannungszustand in ebenen Verbundkörpern. Z. angew. Math. und Mech., Bd. 39, №5/6, 1959.
10. Carafoli, E., Tragflügeltheorie. Berlin, VEB Verlag Technik, 1954.
11. Shtaerman, I. Ia., Contact Problem of the Theory of Elasticity. M. Gostekhizdat, 1949.
12. Muskhelishvili, N. I., Singular Integral Equations. 2nd Ed., M. Fizmatgiz, 1952.
13. Kantorovich, L. V. and Krylov, V. I., Approximate Methods of Higher Analysis. M.-L., Gostekhizdat, 1950.
14. Sherman, D. I., On bending of a circular plate partially clamped and partially supported on the contour. Dokl. Akad. Nauk SSSR, Vol. 101, №4, 1955.

Translated by B. D.